# Transversely Magnetized Oblate Spheroids

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September 12, 2005

**Abstract**

The use of magnetized spheroids has attracted considerable interest as a building block to model and predict the static magnetic fields around steel structures. Defence R&D Canada (DRDC) has made effective use of magnetized prolate spheroids for the near field magnetic modelling of naval ships as required for mine simulation studies. For some geometries however, oblate spheroids may provide a more realistic representation of the structure. In particular, presented here is the derivation of the scalar magnetic potential and magnetic fields arising from a transversely magnetized oblate spheroid. Additionally, the solution is verified to produce; the results for a sphere, a point dipole in the farfield and lastly a transversely magnetized prolate spheroid.

## 1 Introduction

Modelling of the static magnetic fields of naval ships is required to assess the vessel's vulnerability to sea mines, which exploit this signature for mine detonation. Magnetized prolate spheroids are routinely used by DRDC to model a ship's static magnetic signature and to conduct ship-mine simulation studies. The technique uses a distribution of magnetized prolate spheroids about a volume representing the vessel and magnetization weights are obtained by fitting the model to collected magnetic field data. For some structures an oblate spheroid (or a combination of prolate and oblate spheroids) may provide a more realistic representation of the structure being modelled. In particular, presented here is the derivation of the scalar magnetic potential and magnetic fields arising from a transversely magnetized oblate spheroid. In addition, the solution is verified asymptotically to produce the results for a sphere and a point dipole through the use of the appropriate limits. Lastly the transversely magnetized prolate spheroid solutions are derived using a transformation of the coordinate parameters. Maple worksheets with the solution details are provided as an appendix.

## 1.1 Magnetized spheroids

Both prolate and oblate spheroids are volumes formed by revolving an ellipse around one of its principal axes. Prolate spheroids are egg-like while oblate spheroids are similar to discs (or flying saucers). Their use in magnetic modelling of ships is constrained to the case where the spheroid has a uniform permanent magnetization or where the spheroid's magnetic permeability is constant and the spheroid is placed in a uniform applied magnetic field, the latter is referred to as the induced case. The solutions for the magnetized or induced cases are shown in [1] to differ by only a geometric constant, in other words the induced magnetic field of a spheroid with constant magnetic permeability when placed in a uniform field, is also uniform. In contrast to this, the report shows that the solutions for a magnetization transverse to the axis of revolution must be treated separately from that when the magnetization is longitudinal to the axis of revolution. The solution for an oblate spheroid with a transverse magnetization is derived here, the longitudinally magnetized oblate spheroid although not presented can be derived in a similar manner.

## 1.2 Magnetostatic review

Under the condition of a steady magnetic state and assuming that no electrical currents are present Maxwell's equations simplify to the magnetostatic case (see [2] for a more complete description of magnetostatics) described by the following two vector field equations

$$\boldsymbol{\nabla} \times \boldsymbol{H} = 0 \tag{1}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{2}$$

where $\boldsymbol{H}$ is the magnetic field intensity in (A/m), $\boldsymbol{B}$ is the magnetic field in (T), and the curl ($\boldsymbol{\nabla}\times$) and divergence ($\boldsymbol{\nabla}\cdot$) operators are defined in their usual manner. Any vector field with a curl of zero can be expressed as the gradient of a scalar potential and therefore the magnetic intensity can be written as

$$\boldsymbol{H} = -\boldsymbol{\nabla}\psi \tag{3}$$

where $\boldsymbol{\nabla}$ is the gradient operator and $\psi$ is the scalar magnetic potential.

Magnetized spheroids can be treated as permanent magnets with a magnetization $\boldsymbol{M}$ existing inside the spheroid, while outside the spheroid the magnetization drops to zero. Inside the spheroid the magnetization is related to the magnetic field and magnetic intensity by

$$\boldsymbol{B}_0 = \mu_0 \left( \boldsymbol{H}_0 + \boldsymbol{M} \right) \tag{4}$$

where $\mu_0$ is the magnetic permeability of free space and a 0 (1) subscript is used to denote field values inside(outside) the spheroid. Observe that the magnetization has the same units as the magnetic intensity. Substitution of (4) into (2) and using $\boldsymbol{H}_0 = -\boldsymbol{\nabla}\psi_0$ yields

$$\nabla^2 \psi_0 = -\boldsymbol{\nabla} \cdot \boldsymbol{M} \tag{5}$$

where $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is the Laplacian operator. When $\boldsymbol{M}$ is uniform as considered here $\boldsymbol{\nabla} \cdot \boldsymbol{M} = 0$ (except at the spheroid surface) and

$$\nabla^2 \psi_0 = 0 \tag{6}$$

which is known as Laplace's equation for magnetostatics. On the surface of the spheroid there is a an abrupt change in $\boldsymbol{M}$ and (5) must be used. Outside of the spheroid $\boldsymbol{B}_1 = \mu_0 \boldsymbol{H}_1$ since a constant magnetic permeability is assumed and we have $\nabla^2 \psi_1 = 0$.

## 1.3 Oblate spheroidal spheroidal coordinates

Problems involving the magnetic fields of an oblate spheroid are best solved by converting to an oblate spheroidal coordinate system, such as the one shown in Figure (1). For this system the cartesian coordinates $(x, y, z)$ map to the oblate spheroidal coordinates $(\xi, \eta, \phi)$ using the following equations

$$x = c\sqrt{(\xi^2+1)(1-\eta^2)}\cos(\phi) \tag{7}$$

$$y = c\sqrt{(\xi^2+1)(1-\eta^2)}\sin(\phi) \tag{8}$$

$$z = c\,\xi\,\eta, \tag{9}$$

where $c = \sqrt{b^2 - a^2}$ is the focal length of the ellipse and $a$ and $b$ are the semi-minor and semi-major radii of the spheroid with $a < b$. Note the formation of the oblate spheroid as the ellipse revolves around the $z$-axis. Through some rather tedious manipulation[1] the inverse mapping is expressed as

$$\xi = \frac{1}{2c}\left(\sqrt{x^2+y^2+(z-jc)^2} + \sqrt{x^2+y^2+(z+jc)^2}\right) \tag{10}$$

$$\eta = \frac{j}{2c}\left(\sqrt{x^2+y^2+(z-jc)^2} - \sqrt{x^2+y^2+(z+jc)^2}\right) \tag{11}$$

$$\phi = \arctan\left(\frac{y}{x}\right). \tag{12}$$

where $j$ is the imaginary unit. It can be shown that the oblate spheroidal coordinates are real valued and defined on the following ranges

$$0 \le \xi < \infty \qquad -1 \le \eta \le 1 \qquad 0 \le \phi \le 2\pi. \tag{13}$$

When $\xi = \xi_0$ a constant, the surface of an oblate spheroid is generated and the radii of the oblate spheroid can be expressed as $a = c\,\xi_0$ and $b = c\sqrt{\xi_0^2 + 1}$. The volume of the oblate spheroid is given as

$$v = \frac{4\pi}{3} a\, b^2 \tag{14}$$

$$= \frac{4\pi}{3} c^3 \xi_0 \left(\xi_0^2 + 1\right). \tag{15}$$

[1] Maple's `MapToBasis` function arrives at an equivalent but different form of these equations.

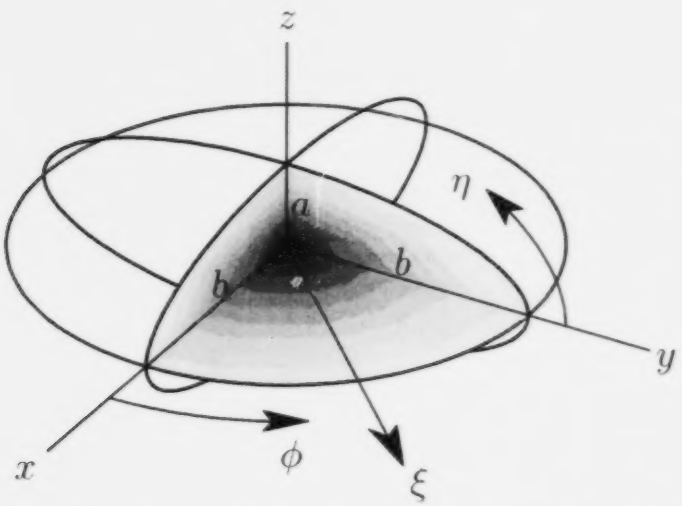



Figure 1: Oblate spheroidal coordinates $(\xi, \eta, \phi)$ in relation to cartesian coordinates $(x, y, z)$.

## 1.4 Transversely magnetized spheroid

We impose that the spheroid possesses a constant magnetization in the $x$-direction which is transverse to the axis of revolution. This magnetization can then be described as

$$\boldsymbol{M} = M_x \boldsymbol{e}_x \tag{16}$$

where $M_x$ is constant and $\boldsymbol{e}_x$ is the unit vector in the $x$-direction. Inside the spheroid because the magnetization is constant the scalar magnetic potential will vary linearly with the $x$ position and can be represented as

$$\psi_0 = d\, M_x\, x \tag{17}$$

where $d$ is the transverse demagnetizing factor, a dimensionless constant which depends on the spheroid geometry. The magnetic field intensity inside the spheroid is using (3) given as

$$\boldsymbol{H}_0 = -d\, M_x\, \boldsymbol{e}_x \tag{18}$$

and for the magnetic field

$$\begin{aligned} \boldsymbol{B}_0 &= \mu_0 (1 - d)\, M_x\, \boldsymbol{e}_x & (19) \\ &= \mu_0 \left(1 - \frac{1}{d}\right) \boldsymbol{H}_0. & (20) \end{aligned}$$

Any solution must also satisfy the boundary conditions for the magnetic field and magnetic intensity such that the component of $\boldsymbol{B}$ normal to the boundary is conserved while the tangential component of $\boldsymbol{H}$ is conserved.

It is also worth introducing the magnetic moment $\boldsymbol{m}$ in (A m$^2$) which is defined as the product of the magnetization and the volume of the magnetic material $v$ in (m$^3$) such that $\boldsymbol{m} = \boldsymbol{M}\, v$. Both the magnetization and dipole moment are vector quantities.

## 2 Solution of Laplace's equation in oblate spheroidal coordinates

Outside of the oblate spheroid the scalar magnetic potential obeys Laplace's equation which in oblate spheroidal coordinates can be expanded as

$$\left(\xi^2+1\right)\frac{\partial^2\psi_1}{\partial\xi^2}+2\xi\frac{\partial\psi_1}{\partial\xi}+\left(1-\eta^2\right)\frac{\partial^2\psi_1}{\partial\eta^2}-2\eta\frac{\partial\psi_1}{\partial\eta}-\frac{\xi^2+\eta^2}{\left(\xi^2+1\right)\left(1-\eta^2\right)}\frac{\partial^2\psi_1}{\partial\phi^2}=0. \quad (21)$$

If $\psi_1$ is separated into the product of three functions, each a function of a single oblate spheroidal coordinate i.e $\psi_1 = R(\xi)T(\eta)P(\phi)$, (21) will separate into the following ordinary differential equations

$$\left(\xi^2+1\right)\frac{d^2R}{d\xi^2}+2\xi\frac{dR}{d\xi}+\left(\frac{\mathrm{m}^2}{\left(\xi^2+1\right)}+\mathrm{k}\right)R = 0 \quad (22)$$

$$\left(1-\eta^2\right)\frac{d^2T}{d\eta^2}-2\eta\frac{dT}{d\eta}+\left(\frac{\mathrm{m}^2}{\left(1-\eta^2\right)}+\mathrm{k}\right)T = 0 \quad (23)$$

$$\frac{d^2P}{d\phi^2}+\mathrm{m}^2P = 0. \quad (24)$$

Where k and m are introduced as separation constants (see [3, p420] for a reference using prolate spheroidal coordinates). Defining $\mathrm{k} = -\mathrm{n}(\mathrm{n}-1)$ the general solutions for these equations are known and given as

$$R = a_{\mathrm{m,n}}P_{\mathrm{n}}^{\mathrm{m}}(j\xi)+b_{\mathrm{m,n}}Q_{\mathrm{n}}^{\mathrm{m}}(j\xi) \quad (25)$$

$$T = c_{\mathrm{m,n}}P_{\mathrm{n}}^{\mathrm{m}}(\eta)+d_{m,n}Q_{\mathrm{n}}^{\mathrm{m}}(\eta) \quad (26)$$

$$P = e_{\mathrm{m}}\sin(\mathrm{m}\phi)+f_{\mathrm{m}}\cos(\mathrm{m}\phi) \quad (27)$$

where $P_{\mathrm{n}}^{\mathrm{m}}$ and $Q_{\mathrm{n}}^{\mathrm{m}}$ are the associated Legendre functions of the 1st and 2nd kind[2] respectively of order m and degree n and $a_{m,n}, b_{\mathrm{m,n}}, c_{\mathrm{m,n}}, d_{\mathrm{m,n}}, e_{\mathrm{m}}$ and $f_{\mathrm{m}}$ are constants to be determined to match the boundary conditions and satisfy finiteness of the solution. Since our solution for the magnetic potential must be finite over all space $d_{\mathrm{m,n}} = a_{\mathrm{m,n}} = 0$ and recombining the constants the general solution for the magnetic potential outside the spheroid is

$$\psi_1 = \sum_{\mathrm{n}}\sum_{\mathrm{m}} b_{\mathrm{m,n}}Q_{\mathrm{n}}^{\mathrm{m}}(j\xi)\,P_{\mathrm{n}}^{\mathrm{m}}(\eta)\left(e_{\mathrm{m}}\sin(m\phi)+f_{\mathrm{m}}\cos(m\phi)\right). \quad (28)$$

### 2.1 Matching the boundary conditions

First the magnetic field intensity inside the oblate spheroid is mapped onto the oblate spheroidal coordinate system yielding

$$\boldsymbol{H}_0 = -\frac{d\,M_x\,\xi\sqrt{(1-\eta^2)}\cos(\phi)}{\sqrt{(\xi^2+\eta^2)}}\boldsymbol{e}_\xi+\frac{d\,M_x\,\eta\sqrt{(\xi^2+1)}\cos(\phi)}{\sqrt{(\xi^2+\eta^2)}}\boldsymbol{e}_\eta+d\,M_x\sin(\phi)\,\boldsymbol{e}_\phi. \quad (29)$$

[2]These definitions are in accordance with those used in Maple 9.5

Outside of the spheroid the magnetic intensity is computed as the negative gradient of the potential $\psi_1$. To match the boundary conditions an investigation of the Legendre functions reveals $m = 1$, $n = 1$ and $f_1 = 0$. For the tangential component of $\boldsymbol{H}$

$$H_{0_\eta} = H_{1_\eta} \tag{30}$$

and matching the normal component of $\boldsymbol{B}$

$$\left(1 - \frac{1}{d}\right) H_{0_\xi} = H_{1_\xi}. \tag{31}$$

Solving the simultaneous set equations yields the solution for the demagnetizing factor as

$$d = \frac{1}{4}\xi_0 \left(\xi_0^2 + 1\right) \arctan\left(2\xi_0, \xi_0^2 - 1\right) - \frac{1}{2}\xi_0^2 \tag{32}$$

where arctan is the two-argument inverse of the tangent function and inspite of the singularity at $\xi = 1$ is well behaved for all $\xi > 0$. The potential outside of the oblate spheroid is also determined and is given as

$$\psi_1 = \frac{1}{2}\xi_0 \left(\xi_0^2 + 1\right)\, c\, M_x \left(\frac{1}{2}\sqrt{\xi^2 + 1} \arctan\left(2\xi, \xi^2 - 1\right) - \frac{\xi}{\sqrt{\xi^2 + 1}}\right) \sqrt{1 - \eta^2}\, \sin(\phi)\,. \tag{33}$$

The magnetic intensity is obtained, as usual, as the negative gradient of this potential. The reader is referred to the appendix for these equations. It is noted that the cartesian components of $\boldsymbol{H}_1$ are derived in terms of the oblate spheroidal coordinates. Thus given an $(x, y, z)$ cartesian location the oblate spheroidal coordinates are computed using (10)-(12) and then substituted into the equations for the potential and magnetic intensity. Figure(2) provides a pictorial representation of the scalar magnetic potential and the magnetic intensity.

# 3 Solution verification

## 3.1 Uniformly magnetized sphere

As $a \to b$ the oblate spheroid will become a sphere and the solution should simplify to that of a magnetic dipole as described by [2, p194]. We look first at how the oblate spheroidal coordinates are transformed to spherical coordinates. In the limit as $a \to b$, $c \to 0$ but $c\,\xi \to r$ remains finite. Further substitution that $\eta = \cos(\theta)$ leads to the (standard) spherical coordinate system

$$x = r \sin(\theta) \cos(\phi) \tag{34}$$

$$y = r \sin(\theta) \sin(\phi) \tag{35}$$

$$z = r \cos(\theta) \tag{36}$$

which justifies our original choice of the left-handed oblate spheroidal coordinate system (see Figure (1)) since $(\xi, \eta. \phi) \to (r, \theta, \phi)$.

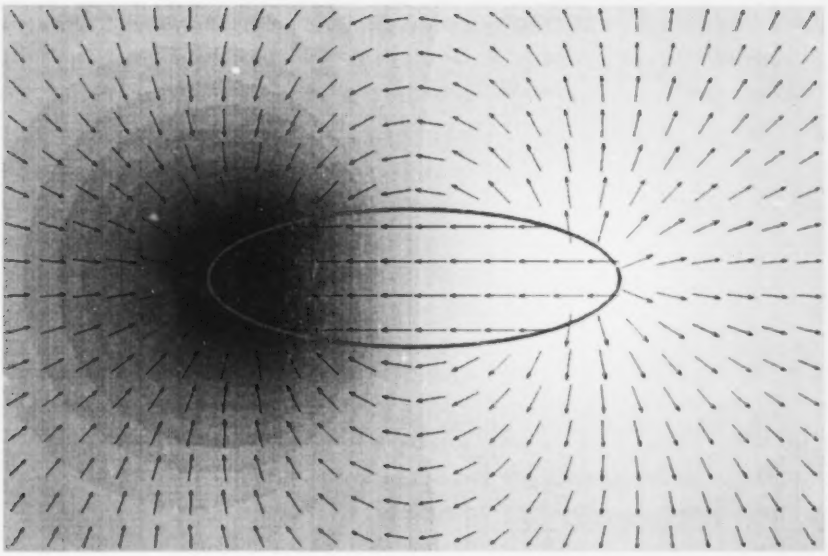


Figure 2: Magnetic scalar potential and magnetic intensity of a transversely magnetized oblate spheroid on the $x-z$ plane. The scalar magnetic potential is represented using a colour intensity and the magnetic intensity is represented using logarithmically scaled quivers.

Inside the sphere, we can apply $\xi_0 \to \infty$ to determine that the demagnetizing factor converges to $1/3$ and that the scalar magnetic potential becomes

$$\psi_0 = \frac{1}{3} M_x \, x \tag{37}$$

as required. For the scalar magnetic potential outside the sphere, the term in $\xi$ is expanded as a series with $c\,\xi \to r$ and with $M_x = m_x/v$ the scalar potential becomes

$$\psi_1 = \frac{m_x}{4\pi r^2} \sin\theta \cos\phi, \tag{38}$$

that of an $x$-oriented magnetic dipole in the standard spherical coordinate presented above. The potential outside of a sphere with uniform magnetization is precisely that of a magnetic dipole.

## 3.2 Point dipole in the farfield

At distances much greater than the size of the spheroid, we have $\xi \gg 1$ and with $c\xi \to r$ and $\eta = \cos(\theta)$ as before the oblate spheroidal coordinates are approximated by the spherical coordinates above. Applying the same expansion of the $\xi$ term and substituting $M_x = m_x/v$ yields the result in (38) as required.

### 3.3 Transversely magnetized prolate spheroid

To obtain the solutions for the prolate spheroid the following transformation of the oblate spheroidal coordinates can be made [4, 5]

$$c \rightarrow -jc \tag{39}$$

$$\xi \rightarrow j\xi. \tag{40}$$

Applying this transformation of variables the mapping from prolate spheroidal coordinates $(\xi, \eta, \phi)$ to cartesian coordinates $(x, y, z)$ is

$$x = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos(\phi) \tag{41}$$

$$y = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin(\phi) \tag{42}$$

$$z = c\,\xi\eta \tag{43}$$

where $\xi > 1$ is required. The transverse demagnetizing factor transforms to

$$d = -\frac{1}{4}\xi_0\left(\xi_0^2 - 1\right)\ln\left(\frac{\xi_0 + 1}{\xi_0 - 1}\right) + \frac{1}{2}\xi_0^2 \tag{44}$$

while the scalar magnetic potential outside the spheroid becomes

$$\psi_1 = \frac{1}{2}\xi_0\left(\xi_0^2 - 1\right)\, c\, M_x \xi_0 \left(\frac{\xi}{\sqrt{\xi^2 - 1}} - \frac{1}{2}\sqrt{\xi^2 - 1}\ln\left(\frac{\xi + 1}{\xi - 1}\right)\right)\sqrt{1 - \eta^2}\cos(\phi) \tag{45}$$

which are the same results derived in [1].

## 4 Conclusions

The scalar magnetic potential, magnetic field and magnetic intensity of an oblate spheroid with a uniform transverse magnetization were derived. The results were verified to converge correctly for a sphere and a point dipole at large distances. The transverse magnetized prolate spheroid solution was also determined by using a transformation of the coordinate parameters.

The use of Maple to derive these results does not eliminate the need to understand the underlying physics of the problem but greatly eases the algebraic burden required for the derivation.

**Keywords:**
magnetic modeling, static magnetic model, prolate spheroid, oblate spheroid, prolate spheroidal coordinates, oblate spheroidal coordinates, transversely magnetized

## References

[1] Holtham, Peter M. and Lucas, Carmen E. (1993). New Approaches to Magnetic Modelling I. Prolate Spheroids II. One-Spike-at-a-Time Fitting. (Technical Report 93-81). Defence Research Establishment Pacific.

[2] Jackson, John David (1975). Classical Electrodynamics, 2 ed. John Wiley and Sons.

[3] Stratton, Julius Adams (1941). Electromagnetic Theory, McGraw-Hill Publishing Company.

[4] Falloon, Peter (2001). Hybrid Numeric/Symbolic Computation of the Spheroidal Harmonics and application to the Generalised Hydrogen Molecular Ion Problem. Master's thesis. University of Western Austrailia.

[5] Barrowes, Benjamin Earl (2004). Electromagnetic Scattering and Induction Models for Spheroidal Geometries. Ph.D. thesis. Massachusetts Institute of Technology.

# Transversely Magnetized Oblate Spheroid: by Troy Richards

```
> restart;
```

```
> with(VectorCalculus): with(plots):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, series
Warning, the name changecoords has been redefined
```

## define oblate spheroidal coordinates:

```
> assume(xi>=0,eta<=1,eta>=-1,phi>0,phi<2*Pi,c>0);
```

```
> eq_x := x = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*cos(phi);
```

$$eq_x := x = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\cos(\varphi)$$

```
> eq_y := y = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*sin(phi);
```

$$eq_y := y = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\sin(\varphi)$$

```
> eq_z := z = c*eta*xi;
```

$$eq_z := z = c\,\eta\,\xi$$

```
> AddCoordinates(oblatespheroidal[xi,eta,phi],[rhs(eq_x),rhs(e
  _y),rhs(eq_z)],overwrite);
```

$$oblatespheroidal$$

```
> SetCoordinates(oblatespheroidal[xi,eta,phi]);
```

$$oblatespheroidal_{\xi,\,\eta,\,\varphi}$$

```
> v := <xi,eta,phi>;
```

$$v := \xi\,e_\xi + \eta\,e_\eta + \varphi\,e_\varphi$$

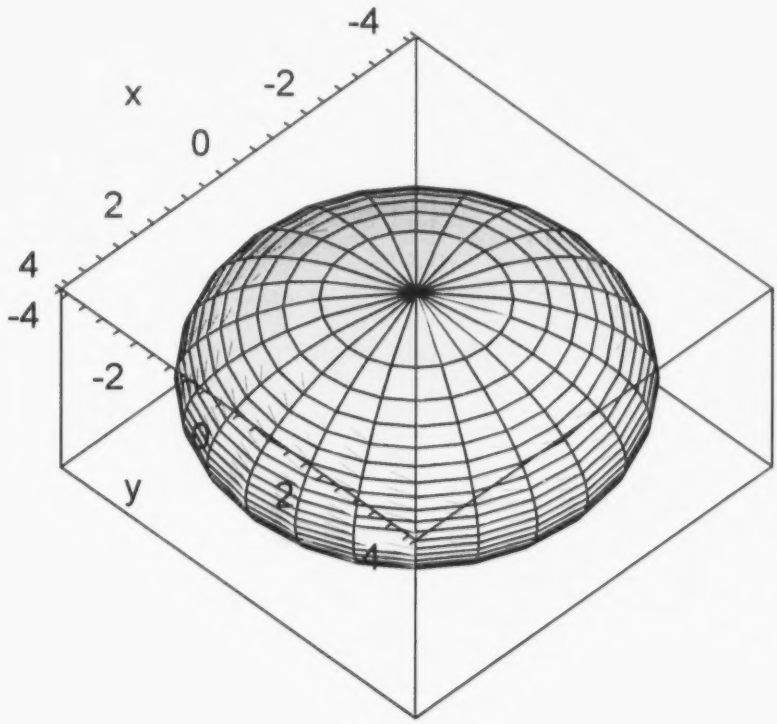
```
> vc := MapToBasis(v,cartesian[x,y,z]);
```

$$vc := c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\cos(\varphi)\,e_x + c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\sin(\varphi)\,e_y + c\,\eta\,\xi\,e_z$$

```
> vp := simplify(MapToBasis(vc,oblatespheroidal[xi,eta,phi]));
```

$$vp := \xi\,e_\xi + \eta\,e_\eta + \arctan(\sin(\varphi),\cos(\varphi))\,e_\varphi$$

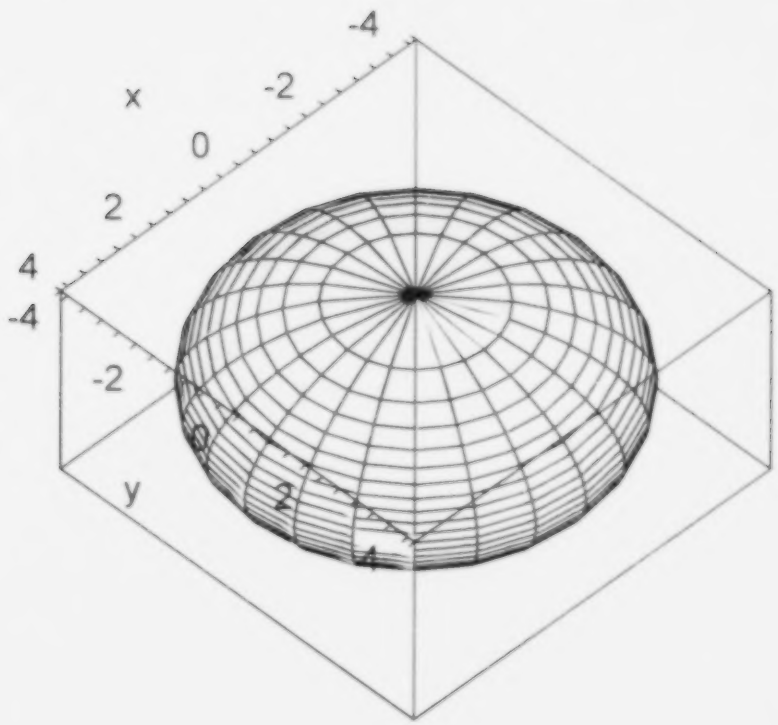
```
> vv := unapply(vc,xi,eta,phi,c):
> plot3d(subs(xi0=a/c,c=sqrt(b^2-a^2),a=2,b=4,vv(xi0,eta,phi,c
  ),
  eta=-1..1,phi=0..2*Pi,axes=boxed,labels=[x, y,
  z],scaling=constrained);
```




## solve Laplace's equation in oblate spheroidal coordinates

```
> alias(psi=psi(xi,eta,phi)):
> pde1 := Laplacian(psi)=0;
```

```
> vv := unapply(vc,xi,eta,phi,c):
> plot3d(subs(xi0=a/c,c=sqrt(b^2-a^2),a=2,b=4,vv(xi0,eta,phi,c
  ),
  eta=-1..1,phi=0..2*Pi,axes=boxed,labels=[x, y,
  z],scaling=constrained);
```




## solve Laplace's equation in oblate spheroidal coordinates

```
> alias(psi=psi(xi,eta,phi)):
> pde1 := Laplacian(psi)=0:
```

$$pde1 := \frac{1}{c^2(\eta^2+\xi^2)}\left(2c\xi\left(\frac{\partial}{\partial\xi}\psi\right) + c(\xi^2+1)\left(\frac{\partial^2}{\partial\xi^2}\psi\right) - 2c\eta\left(\frac{\partial}{\partial\eta}\psi\right) + c(-\eta^2+1)\left(\frac{\partial^2}{\partial\eta^2}\psi\right) + \frac{c(\eta^2+\xi^2)\left(\frac{\partial^2}{\partial\varphi^2}\psi\right)}{(\xi^2+1)(-\eta^2+1)}\right) = 0$$

```
> pde1 := expand((pde1*c^2)*(xi^2+eta^2)):
```

```
> pde1 := collect(pde1,[diff(psi,xi,xi), diff(psi,xi),
  diff(psi,eta,eta),
  diff(psi,eta),diff(psi,phi,phi)],simplify);
```

$$pde1 := (\xi^2+1)\left(\frac{\partial^2}{\partial\xi^2}\psi\right) + 2\xi\left(\frac{\partial}{\partial\xi}\psi\right) + (-\eta^2+1)\left(\frac{\partial^2}{\partial\eta^2}\psi\right) - 2\eta\left(\frac{\partial}{\partial\eta}\psi\right) - \frac{(\eta^2+\xi^2)\left(\frac{\partial^2}{\partial\varphi^2}\psi\right)}{(\xi^2+1)(\eta^2-1)} = 0$$

## see Stratton p421 for separation of variables

```
> alias(R=R(xi),T=T(eta),P=P(phi));
```

$$\psi, R, T, P$$

```
> vars := [diff(R,xi,xi), diff(R,xi), diff(T,eta,eta),
  diff(T,eta), diff(P,phi,phi)];
```

$$vars := \left[\frac{\partial^2}{\partial\xi^2}R, \frac{\partial}{\partial\xi}R, \frac{\partial^2}{\partial\eta^2}T, \frac{\partial}{\partial\eta}T, \frac{\partial^2}{\partial\varphi^2}P\right]$$

```
> pde2 :=
  collect(expand(subs(psi=R*T*P,pde1/(R*T*P))),vars,simplify);
```

$$pde2 := \frac{(\xi^2+1)\left(\frac{\partial^2}{\partial \xi^2} R\right)}{R} + \frac{2\xi\left(\frac{\partial}{\partial \xi} R\right)}{R} - \frac{(\eta^2-1)\left(\frac{\partial^2}{\partial \eta^2} T\right)}{T} - \frac{2\eta\left(\frac{\partial}{\partial \eta} T\right)}{T} - \frac{(\eta^2+\xi^2)\left(\frac{\partial^2}{\partial \varphi^2} P\right)}{P(\xi^2+1)(\eta^2-1)} = 0$$

```
> kk := (xi^2+eta^2)/(xi^2+1)/(1-eta^2);
```

$$kk := \frac{\eta^2+\xi^2}{(\xi^2+1)(-\eta^2+1)}$$

```
> pde2 := collect(pde2/kk,vars,simplify);
```

$$pde2 := -\frac{(\xi^2+1)^2(\eta^2-1)\left(\frac{\partial^2}{\partial \xi^2} R\right)}{(\eta^2+\xi^2)R} - \frac{2(\xi^2+1)^2(\eta^2-1)\xi\left(\frac{\partial}{\partial \xi} R\right)}{(\eta^2+\xi^2)R} + \frac{(\xi^2+1)^2(\eta^2-1)^2\left(\frac{\partial^2}{\partial \eta^2} T\right)}{(\eta^2+\xi^2)T} + \frac{2(\xi^2+1)^2(\eta^2-1)\eta\left(\frac{\partial}{\partial \eta} T\right)}{(\eta^2+\xi^2)T} + \frac{\frac{\partial^2}{\partial \varphi^2} P}{P} =$$

```
> ode1 := diff(P,phi,phi)/P = -m^2;
```

$$ode1 := \frac{\frac{\partial^2}{\partial \varphi^2} P}{P} = -m^2$$

```
> pde3 := subs(ode1,pde2);
```

$$pde3 := -\frac{(\xi^2+1)^2(\eta^2-1)\left(\frac{\partial^2}{\partial \xi^2} R\right)}{(\eta^2+\xi^2)R} - \frac{2(\xi^2+1)(\eta^2-1)\xi\left(\frac{\partial}{\partial \xi} R\right)}{(\eta^2+\xi^2)R}$$

$$+\frac{(\xi^2+1)(\eta^2-1)^2\left(\frac{\partial^2}{\partial\eta^2}T\right)}{(\eta^2+\xi^2)T}+\frac{2(\xi^2+1)(\eta^2-1)\eta\left(\frac{\partial}{\partial\eta}T\right)}{(\eta^2+\xi^2)T}-m^2=0$$

```
> pde3 := collect(pde3*kk,vars,simplify);
```

$$pde3:=\frac{(\xi^2+1)\left(\frac{\partial^2}{\partial\xi^2}R\right)}{R}+\frac{2\xi\left(\frac{\partial}{\partial\xi}R\right)}{R}-\frac{(\eta^2-1)\left(\frac{\partial^2}{\partial\eta^2}T\right)}{T}-\frac{2\eta\left(\frac{\partial}{\partial\eta}T\right)}{T}+\frac{(\eta^2+\xi^2)m^2}{(\xi^2+1)(\eta^2-1)}=0$$

```
> pde3 :=
lhs(pde3)-coeff(lhs(pde3),m,2)*m^2=rhs(pde3)-coeff(lhs(pde3)
m,2)*m^2;
```

$$pde3:=\frac{(\xi^2+1)\left(\frac{\partial^2}{\partial\xi^2}R\right)}{R}+\frac{2\xi\left(\frac{\partial}{\partial\xi}R\right)}{R}-\frac{(\eta^2-1)\left(\frac{\partial^2}{\partial\eta^2}T\right)}{T}-\frac{2\eta\left(\frac{\partial}{\partial\eta}T\right)}{T}=\frac{(\eta^2+\xi^2)m^2}{(\xi^2+1)(\eta^2-1)}$$

```
> pde3 := lhs(pde3)-convert(rhs(pde3),parfrac,xi,true)=0;
```

$$pde3:=\frac{(\xi^2+1)\left(\frac{\partial^2}{\partial\xi^2}R\right)}{R}+\frac{2\xi\left(\frac{\partial}{\partial\xi}R\right)}{R}-\frac{(\eta^2-1)\left(\frac{\partial^2}{\partial\eta^2}T\right)}{T}-\frac{2\eta\left(\frac{\partial}{\partial\eta}T\right)}{T}+\frac{m^2}{\eta^2-1}+\frac{m^2}{\xi^2+1}=0$$

```
> gg := subs([diff(R,xi,xi)=0,diff(R,xi)=0],lhs(pde3)):
> gg := simplify(limit(gg,xi=infinity)):
> ode2 := collect(gg,vars,simplify) = k;
```

$$ode2 := -\frac{(\eta^2 - 1)\left(\frac{\partial^2}{\partial \eta^2} T\right)}{T} - \frac{2\eta\left(\frac{\partial}{\partial \eta} T\right)}{T} + \frac{m^2}{\eta^2 - 1} = k$$

```
> ode3 := pde3 - ode2;
```

$$ode3 := \frac{(\xi^2 + 1)\left(\frac{\partial^2}{\partial \xi^2} R\right)}{R} + \frac{2\xi\left(\frac{\partial}{\partial \xi} R\right)}{R} + \frac{m^2}{\xi^2 + 1} = -k$$

```
> ode2 :=
  collect(expand(T*(lhs(ode2)-rhs(ode2))),vars,simplify)=0;
```

$$ode2 := (-\eta^2 + 1)\left(\frac{\partial^2}{\partial \eta^2} T\right) - 2\eta\left(\frac{\partial}{\partial \eta} T\right) - \frac{T(k\eta^2 - k - m^2)}{\eta^2 - 1} = 0$$

```
> ode3 :=
  collect(expand(R*(lhs(ode3)-rhs(ode3))),vars,simplify)=0;
```

$$ode3 := (\xi^2 + 1)\left(\frac{\partial^2}{\partial \xi^2} R\right) + 2\xi\left(\frac{\partial}{\partial \xi} R\right) + \frac{R(k\xi^2 + k + m^2)}{\xi^2 + 1} = 0$$

```
> soln1 := dsolve(ode1,P);
```

$$soln1 := P = _C1 \sin(m\varphi) + _C2 \cos(m\varphi)$$

```
> soln2 := dsolve(ode2,T);
```

$$soln2 := T = _C1\,\mathrm{LegendreP}\left(\frac{1}{2}\sqrt{-4k+1} - \frac{1}{2}, m, \eta\right) + _C2\,\mathrm{LegendreQ}\left(\frac{1}{2}\sqrt{-4k+1} - \frac{1}{2}, m, \eta\right)$$

```
> soln3 := dsolve(ode3,R);
```

$$soln3 := R = _C1\,\mathrm{LegendreP}\left(\frac{1}{2}\sqrt{-4k+1} - \frac{1}{2}, m, I\xi\right) + _C2\,\mathrm{LegendreQ}\left(\frac{1}{2}\sqrt{-4k+1} - \frac{1}{2}, m, I\xi\right)$$

```
> assume(n>0);
```

```
> soln1 := subs(_C1=e[m],_C2=f[m],soln1);
```

$$soln1 := P = e_m \sin(m\,\varphi) + f_m \cos(m\,\varphi)$$

```
> soln2 := simplify(subs(k=-n*(n+1),_C1=1,_C2=0,soln2));
```

$$soln2 := T = \text{LegendreP}(n, m, \eta)$$

```
> soln3 := simplify(subs(k=-n*(n+1),_C1=0,_C2=b[m,n],soln3));
```

$$soln3 := R = b_{m,\,n}\,\text{LegendreQ}(n, m, I\,\xi)$$

```
> soln := psi = subs(soln1,soln2,soln3,R*T*P);
```

$$soln := \psi = b_{m,\,n}\,\text{LegendreQ}(n, m, I\,\xi)\,\text{LegendreP}(n, m, \eta)\,(e_m \sin(m\,\varphi) + f_m \cos(m\,\varphi))$$

# determine general equation of the magnetic potential outside the spheroid

For a uniformly magnetized *flying saucer* in the x-dirn, where d is the demagnetizing factor and Mx is the magnetization

```
> psi0 := d*Mx*x;
```

$$\psi 0 := d\,Mx\,x$$

inside the *flying saucer* the magnetic intensity is:

```
> SetCoordinates('cartesian'[x,y,z]);
```

$$cartesian_{x,\,y,\,z}$$

```
> H0c := -Gradient(psi0);
```

$$H0c := -d\,Mx\,\underline{e}_x$$

```
> H0 := simplify(MapToBasis(H0c,oblatespheroidal[xi,eta,phi]))
```

$$H0 := -\frac{d\,Mx\cos(\varphi)\sqrt{-\eta^2+1}\;\xi}{\sqrt{\eta^2+\xi^2}}\,\underline{e}_\xi + \frac{d\,Mx\cos(\varphi)\,\eta\sqrt{\xi^2+1}}{\sqrt{\eta^2+\xi^2}}\,\underline{e}_\eta + d\,Mx\sin(\varphi)\,\underline{e}_\varphi$$

Define some grouping functions:

```
> rr := xx-> combine(xx,radical,symbolic):
> ll := xx-> combine(xx,ln):
> tt := xx-> combine(xx,trig):
> pp := xx-> convert(xx,parfrac,xi,true):
> ss := xx-> tt(simplify(xx)):
```

```
> ff := xx->
  collect(simplify(xx),[eta,xi,phi],distributive,ss):
```

Outside of the *flying saucer* the potential is given as (and noting m=1 and n=1):

```
> soln1 := subs([m=1,n=1],soln1);
```

$$soln1 := P = e_1 \sin(\varphi) + f_1 \cos(\varphi)$$

```
> soln2 := simplify(rr(expand(subs([m=1,n=1],soln2))));
```

$$soln2 := T = I\sqrt{-\eta^2 + 1}$$

```
> soln3 := subs([m=1,n=1],soln3);
```

$$soln3 := R = b_{1,1}\,\text{LegendreQ}(1, 1, I\,\xi)$$

```
> soln2 := lhs(soln2) = rhs(soln2)/I;
```

$$soln2 := T = \sqrt{-\eta^2 + 1}$$

```
> soln3 := lhs(soln3) = rhs(soln3)/b[1,1];
```

$$soln3 := R = \text{LegendreQ}(1, 1, I\,\xi)$$

```
> soln3 := subs(xi=Xi/I,soln3);
```

$$soln3 := R(-I\,\Xi) = \text{LegendreQ}(1, 1, \Xi)$$

```
> assume(Xi>0);
> soln3 := ll(simplify(rr(simplify(soln3))));
```

$$soln3 := R(-I\,\Xi) = \frac{1}{2}\left(-\ln\left(\frac{\Xi - 1}{\Xi + 1}\right)\Xi^2 - 2\,\Xi + \ln\left(\frac{\Xi - 1}{\Xi + 1}\right)\right)\sqrt{\frac{1}{\Xi^2 - 1}}$$

```
> soln3 := collect(soln3,ln,simplify);
```

$$soln3 := R(-I\,\Xi) = -\frac{1}{2}(\Xi^2 - 1)\sqrt{\frac{1}{\Xi^2 - 1}}\ln\left(\frac{\Xi - 1}{\Xi + 1}\right) - \Xi\sqrt{\frac{1}{\Xi^2 - 1}}$$

```
> soln3 := subs(Xi=I*xi,soln3);
```

$$soln3 := R = -\frac{1}{2}(-\xi^2 - 1)\sqrt{\frac{1}{-\xi^2 - 1}}\ln\left(\frac{I\,\xi - 1}{I\,\xi + 1}\right) - I\,\xi\sqrt{\frac{1}{-\xi^2 - 1}}$$

```
> soln3 := collect(simplify(evalc(soln3)),arctan);
```

$$soln3 := R = -\frac{1}{2}\sqrt{\xi^2 + 1}\,\arctan(2\,\xi, -1 + \xi^2) + \frac{\xi}{\sqrt{\xi^2 + 1}}$$

```
> assign(soln1,soln2,soln3);
```

```
> psi1 := K*R*T*P;
```

$$\psi 1 := K\left(-\frac{1}{2}\sqrt{\xi^2+1}\,\arctan(2\,\xi,-1+\xi^2)+\frac{\xi}{\sqrt{\xi^2+1}}\right)\sqrt{-\eta^2+1}\,(e_1\sin(\varphi)+f_1\cos(\varphi))$$

## match boundary conditions

```
> SetCoordinates(oblatespheroidal[xi,eta,phi]);
```

$$oblatespheroidal_{\xi,\,\eta,\,\varphi}$$

to match H0 at the boundary must have e1=0 and f1=1:

```
> psi1 := subs(e[1]=0,f[1]=1,psi1);
```

$$\psi 1 := K\left(-\frac{1}{2}\sqrt{\xi^2+1}\,\arctan(2\,\xi,-1+\xi^2)+\frac{\xi}{\sqrt{\xi^2+1}}\right)\sqrt{-\eta^2+1}\,\cos(\varphi)$$

```
> H1 := -Gradient(psi1):
> H1 := rr(simplify(H1)):
> cc := xx-> collect(xx,arctan,simplify):
> H1 := map(cc,H1);
```

$$H1 := \left(\frac{K\,\xi\cos(\varphi)\sqrt{-\eta^2+1}\,\arctan(2\,\xi,-1+\xi^2)}{2\sqrt{\eta^2+\xi^2}\,c}-\frac{(\xi^2+2)\,K\cos(\varphi)\sqrt{-\eta^2+1}}{\sqrt{\eta^2+\xi^2}\,c\,(\xi^2+1)}\right)\underline{e}_{\xi}+\left(-\frac{K\,c\,os(\varphi)\,\eta\sqrt{\xi^2+1}\,\arctan(2\,\xi,-1+\xi^2)}{2\,c\sqrt{\eta^2+\xi^2}}+\frac{K\,\xi\cos(\varphi)\,\eta}{c\sqrt{\eta^2+\xi^2}\sqrt{\xi^2+1}}\right)\underline{e}_{\eta}+\left(-\frac{K\,s\,in(\varphi)\arctan(2\,\xi,-1+\xi^2)}{2\,c}+\frac{K\,\xi\sin(\varphi)}{c\,(\xi^2+1)}\right)\underline{e}_{\varphi}$$

to match the tangential component of H at the boundary:

```
> assume(xi0>0);
> eq1 := (subs(xi=xi0,H0[2]) = subs([xi=xi0],H1[2]));
```

$$eq1 := \frac{d\,Mx\cos(\varphi)\,\eta\sqrt{\xi 0^2+1}}{\sqrt{\eta^2+\xi 0^2}} = -\frac{K\cos(\varphi)\,\eta\sqrt{\xi 0^2+1}\,\arctan(2\,\xi 0,-1+\xi 0^2)}{2\,c\sqrt{\eta^2+\xi 0^2}}$$

$$+ \frac{K\,\xi 0\cos(\varphi)\,\eta}{c\sqrt{\eta^2+\xi 0^2}\,\sqrt{\xi 0^2+1}}$$

```
> eq_K := K = solve(eq1,K);
```

$$eq_K := K = -\frac{2\,d\,Mx\,c\,\xi 0^2 + 2\,d\,Mx\,c}{\arctan(2\,\xi 0, -1+\xi 0^2)\,\xi 0^2 + \arctan(2\,\xi 0, -1+\xi 0^2) - 2\,\xi 0}$$

the normal component of B is conserved across the boundary so:

```
> eq2 := (1-1/d)*subs(xi=xi0,H0[1]) =
  ll(simplify(subs(xi=xi0,eq_K,H1[1])));
```

$$eq2 := -\frac{\left(-\frac{1}{d}+1\right) d\,Mx\cos(\varphi)\sqrt{-\eta^2+1}\;\xi 0}{\sqrt{\eta^2+\xi 0^2}} = -\frac{1}{(\arctan(2\,\xi 0, -1+\xi 0^2)\,\xi 0^2 + \arctan(2\,\xi 0, -1+\xi 0^2) - 2\,\xi 0)\sqrt{\eta^2+\xi 0^2}}(d\,Mx\cos(\varphi$$
$$\sqrt{-\eta^2+1}\,(\xi 0^3\arctan(2\,\xi 0, -1+\xi 0^2) + \xi 0\arctan(2\,\xi 0, -1+\xi 0^2) - 2\,\xi 0^2 - 4))$$

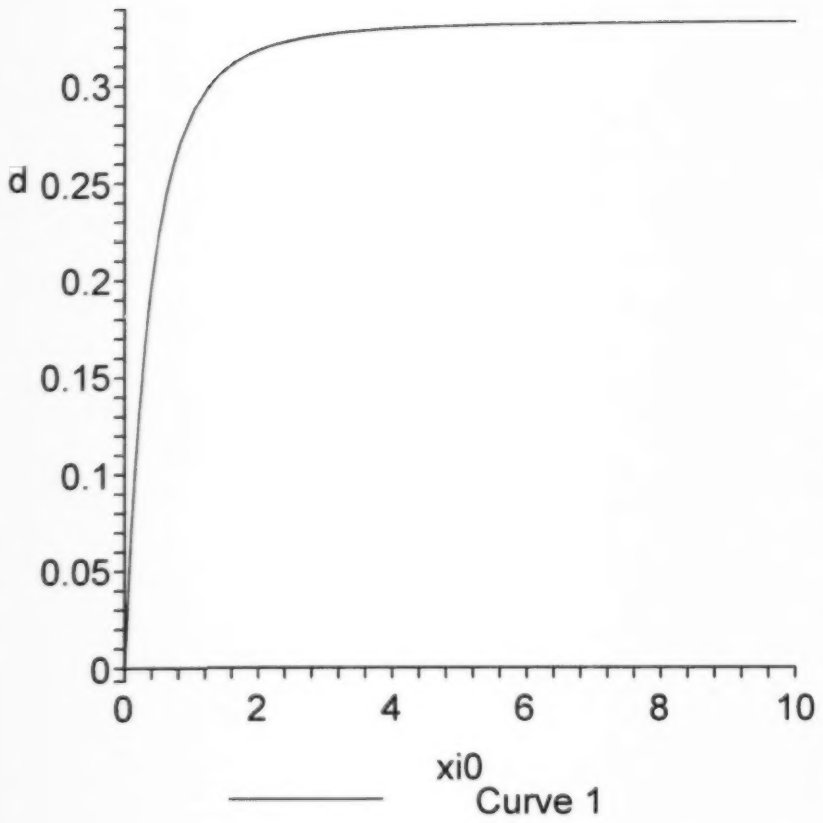
```
> eq_d := d = (solve(eq2,d));
```

$$eq_d := d = \frac{1}{4}\xi 0^3\arctan(2\,\xi 0, -1+\xi 0^2) + \frac{1}{4}\xi 0\arctan(2\,\xi 0, -1+\xi 0^2) - \frac{1}{2}\xi 0^2$$

```
> eq_d := collect(eq_d,arctan,factor);
```

$$eq_d := d = \frac{1}{4}\xi 0\,(\xi 0^2+1)\arctan(2\,\xi 0, -1+\xi 0^2) - \frac{1}{2}\xi 0^2$$

```
> plot(rhs(eq_d),xi0=0..10,labels=['xi0','d']);
```




as eta0->infinity oblate spheroid becomes a spheroid and demagnetizing factor T -> 1/3

```
> limit(rhs(eq_d),xi0=infinity);
```

$$\frac{1}{3}$$

and for the potential inside the spheroid:

```
> limit(subs(eq_d,psi0),xi0=infinity);
```

$$\frac{1}{3} Mx\, x$$

continuning with the solution:

```
> eq_K := simplify(subs(eq_d,eq_K));
```

$$eq_K := K = -\frac{1}{2}(\xi 0^2 + 1)\, c\, Mx\, \xi 0$$

```
> R := subs(eq_K,R);
```

$$R := -\frac{1}{2}\sqrt{\xi^2+1}\ \arctan(2\,\xi, -1+\xi^2) + \frac{\xi}{\sqrt{\xi^2+1}}$$

```
> P := subs(e[1]=0,f[1]=1,P);
```

$$P := \cos(\varphi)$$

```
> assign(eq_K);
```

```
> K := -K; R := -R; T := T; P := P;
```

$$K := \frac{1}{2}(\xi 0^2 + 1)\, c\, Mx\, \xi 0$$

$$R := \frac{1}{2}\sqrt{\xi^2+1}\ \arctan(2\,\xi, -1+\xi^2) - \frac{\xi}{\sqrt{\xi^2+1}}$$

$$T := \sqrt{-\eta^2+1}$$

$$P := \cos(\varphi)$$

```
> psi1 := K*R*T*P;
```

$$\psi 1 := \frac{1}{2}(\xi 0^2 + 1)\, c\, Mx\, \xi 0\left(\frac{1}{2}\sqrt{\xi^2+1}\ \arctan(2\,\xi, -1+\xi^2) - \frac{\xi}{\sqrt{\xi^2+1}}\right)\sqrt{-\eta^2+1}\ \cos(\varphi$$

```
> SetCoordinates('cartesian'[x,y,z]);
```

$$cartesian_{x,\,y,\,z}$$

```
> rr := <-2,4,5>;
```

$$rr := (-2)\,e_x + 4\,e_y + 5\,e_z$$

```
> ro :=
  evalf(simplify(subs(c=sqrt(b^2-a^2),a=2,b=4,MapToBasis(rr,ob
  atespheroidal[xi,eta,phi]))));
```

$$ro := 1.835341645\,e_\xi + 0.7864343280\,e_\eta + 2.034443936\,e_\varphi$$

```
> evalf(subs(xi0=a/c,c=sqrt(b^2-a^2),a=2,b=4,Mx=2000,xi=ro[1],
  ta=ro[2],phi=ro[3],psi1));
```

$$-121.2492921$$

```
>
```

# Magnetic Intnesity from psi:

## Determine Magnetic Intensity from psi:

```
> restart;
```

```
> with(VectorCalculus): with(plots):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, serie
Warning, the name changecoords has been redefined
```

```
> assume(xi>=0,eta<=1,eta>=-1,phi>0,phi<2*Pi,c>0);
```

```
> eq_x := x = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*cos(phi);
```

$$eq_x := x = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\cos(\varphi)$$

```
> eq_y := y = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*sin(phi);
```

$$eq_y := y = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\sin(\varphi)$$

```
> eq_z := z = c*eta*xi;
```

$$eq_z := z = c\,\eta\,\xi$$

```
> AddCoordinates(oblatespheroidal[xi,eta,phi],

  [rhs(eq_x),rhs(eq_y),rhs(eq_z)],overwrite);
```

$$oblatespheroidal$$

```
> SetCoordinates(oblatespheroidal[xi,eta,phi]);
```

$$oblatespheroidal_{\xi,\,\eta,\,\varphi}$$

```
> K := 1/2*(xi0^2+1)*c*Mx*xi0;
```

$$K := \left(\frac{1}{2}\,\xi 0^2 + \frac{1}{2}\right)\,c\,Mx\,\xi 0$$

```
> R := 1/2*(xi^2+1)^(1/2)*arctan(2*xi,
  -1+xi^2)-xi/(xi^2+1)^(1/2);
```

$$R := \frac{1}{2}\sqrt{\xi^2+1}\ \arctan(2\,\xi,\,\xi^2-1) - \frac{\xi}{\sqrt{\xi^2+1}}$$

```
> T := (-eta^2+1)^(1/2);
```

$$T := \sqrt{-\eta^2 + 1}$$

```
> P := cos(phi);
```

$$P := \cos(\varphi)$$

```
> psi1 := R*T*P;
```

$$\psi 1 := \left( \frac{1}{2} \sqrt{\xi^2 + 1}\ \arctan(2\,\xi, \xi^2 - 1) - \frac{\xi}{\sqrt{\xi^2 + 1}} \right) \sqrt{-\eta^2 + 1}\ \cos(\varphi)$$

```
> h1 := -Gradient(psi1):
> aa := xx-> collect(xx,arctan,simplify):
> h1 := map(aa,h1);
```

$$h1 := \left( -\frac{\xi \sqrt{-\eta^2 + 1}\ \cos(\varphi) \arctan(2\,\xi, \xi^2 - 1)}{2\,c \sqrt{\eta^2 + \xi^2}} + \frac{(\xi^2 + 2) \sqrt{-\eta^2 + 1}\ \cos(\varphi)}{(\xi^2 + 1)\,c \sqrt{\eta^2 + \xi^2}} \right) \bar{e}_\xi + \left( \frac{\sqrt{\xi^2 + 1}\ \cos(\varphi)\,\eta \arctan(2\,\xi, \xi^2 - 1)}{2\,c \sqrt{\eta^2 + \xi^2}} - \frac{\xi \cos(\varphi)\,\eta}{c \sqrt{\eta^2 + \xi^2}\ \sqrt{\xi^2 + 1}} \right) \bar{e}_\eta + \left( \frac{\sin(\varphi) \arctan(2\,\xi, \xi^2 - 1)}{2\,c} - \frac{\xi \sin(\varphi)}{c\,(\xi^2 + 1)} \right) \bar{e}_\varphi$$

```
> ax := VectorField(<1,0,0>,cartesian[x,y,z]):
> h1x := DotProduct(ax,h1):
> ay := VectorField(<0,1,0>,cartesian[x,y,z]):
> h1y := DotProduct(ay,h1):
> az := VectorField(<0,0,1>,cartesian[x,y,z]):
> h1z := DotProduct(az,h1):
> h1c := VectorField(<h1x,h1y,h1z>,cartesian[x,y,z]):
> h1c := map(aa,simplify(h1c));
```

$$h1c := \left( -\frac{\arctan(2\,\xi, \xi^2 - 1)}{2\,c} + \frac{\xi\,(-2\cos(\varphi)^2\,\eta^2 + 2\cos(\varphi)^2 + \eta^2 + \xi^2)}{(\eta^2 + \xi^2)\,c\,(\xi^2 + 1)} \right) \bar{e}_x -$$

$$\frac{2\sin(\varphi)\cos(\varphi)\,\xi\,(\eta^2-1)}{(\eta^2+\xi^2)\,c\,(\xi^2+1)}\,\bar{e}_y+\frac{2\,\eta\cos(\varphi)\sqrt{-\eta^2+1}}{\sqrt{\xi^2+1}\,(\eta^2+\xi^2)\,c}\,\bar{e}_z$$

```
> pp := xx-> convert(xx,parfrac,xi,true):
> h1c[1] := pp(h1c[1]):
```

The final form of the solution outside of the flying saucer: (Hx,Hy,Hz) in oblate spheroidal coordinates:

```
> H1c := K*h1c;
```

$$H1c := \left(\frac{1}{2}\xi 0^2+\frac{1}{2}\right)c\,Mx\,\xi 0\left(-\frac{\arctan(2\,\xi,\xi^2-1)}{2\,c}+\frac{2\cos(\varphi)^2\,\xi}{(\eta^2+\xi^2)\,c}-\frac{(2\cos(\varphi)^2-1)\,\xi}{(\xi^2+1)\,c}\right)\bar{e}_x$$
$$\frac{2\left(\frac{1}{2}\xi 0^2+\frac{1}{2}\right)Mx\,\xi 0\sin(\varphi)\cos(\varphi)\,\xi\,(\eta^2-1)}{(\eta^2+\xi^2)\,(\xi^2+1)}\,\bar{e}_y+$$
$$\frac{2\left(\frac{1}{2}\xi 0^2+\frac{1}{2}\right)Mx\,\xi 0\,\eta\cos(\varphi)\sqrt{-\eta^2+1}}{\sqrt{\xi^2+1}\,(\eta^2+\xi^2)}\,\bar{e}_z$$

```
> SetCoordinates('cartesian'[x,y,z]);
```

$$cartesian_{x,\,y,\,z}$$

```
> rr := <-2.,4.,5.>;
```

$$rr := (-2.)\,e_x+4.\,e_y+5.\,e_z$$

```
> ro := MapToBasis(rr,oblatespheroidal[xi,eta,phi]):
> ro := evalf(subs(c=sqrt(b^2-a^2),a=2.,b=4.,ro));
```

$$ro := 1.835341643\,e_\xi+0.7864343281\,e_\eta+2.034443936\,e_\varphi$$

```
> evalf(subs(xi0=a/c,c=sqrt(b^2-a^2),a=2.,b=4.,Mx=2000.,xi=ro[
  ],eta=ro[2],phi=ro[3],H1c));
```

$$(-48.2451884)\,\bar{e}_x-24.75891560\,\bar{e}_y-40.13636547\,\bar{e}_z$$

```
>
```

## cartesian to oblate:

### generate inversion from cartesian to oblate spheroidal coordinates:

```
> restart;
```

```
> with(VectorCalculus): with(plots):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, serie
Warning, the name changecoords has been redefined
```

```
> assume(xi>=0,eta<=1,eta>=-1,phi>0,phi<2*Pi,c>0);
```

```
> eq_x := x = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*cos(phi);
```

$$eq_x := x = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\cos(\varphi)$$

```
> eq_y := y = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*sin(phi);
```

$$eq_y := y = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\sin(\varphi)$$

```
> eq_z := z = c*eta*xi;
```

$$eq_z := z = c\,\eta\,\xi$$

```
> AddCoordinates(oblatespheroidal[xi,eta,phi],

  [rhs(eq_x),rhs(eq_y),rhs(eq_z)],overwrite);
```

$$oblatespheroidal$$

```
> SetCoordinates(oblatespheroidal[xi,eta,phi]);
```

$$oblatespheroidal_{\xi,\,\eta,\,\varphi}$$

```
> v := Vector([x,y,z],'attributes'=['coords'='cartesian']);
```

$$v := x\,e_x + y\,e_y + z\,e_z$$

```
> vp := simplify(MapToBasis(v,'oblatespheroidal'[xi,eta,phi]))
```

```
> vp := map(factor,vp):
```

```
> eq_xi := xi = vp[1];
```

$$eq_xi := \xi = \frac{1}{2c}\left(\mathrm{sqrt}(2z^2 + 2x^2 - 2c^2 + 2y^2 + 2\sqrt{z^4 + 2z^2x^2 + 2c^2z^2 + 2z^2y^2 + x^4 - 2x^2c^2 + 2y^2x^2 + c^4 - 2c^2y^2 + y}\right)$$

```
> eq_eta := eta = vp[2];
```

$$eq_eta := \eta = \frac{(2z)}{\left(\mathrm{sqrt}(2z^2 + 2x^2 - 2c^2 + 2y^2 + 2\sqrt{z^4 + 2z^2x^2 + 2c^2z^2 + 2z^2y^2 + x^4 - 2x^2c^2 + 2y^2x^2 + c^4 - 2c^2y^2 + y}\right)}$$

```
> eq_phi := phi = simplify(vp[3]);
```

$$eq_phi := \varphi = \arctan\left(\frac{y}{\sqrt{y^2 + x^2}}, \frac{x}{\sqrt{y^2 + x^2}}\right)$$

generate cleaner solution:

```
> eq1 := (lhs(eq_z)+I*c)^2 = factor((rhs(eq_z)+I*c))^2;
```

$$eq1 := (z + I\,c)^2 = c^2\,(\eta\,\xi + I)^2$$

```
> eq2 := (lhs(eq_z)-I*c)^2 = factor((rhs(eq_z)-I*c))^2;
```

$$eq2 := (z - I\,c)^2 = c^2\,(\eta\,\xi - I)^2$$

```
> eq3 := lhs(eq_x)^2 = rhs(eq_x)^2;
```

$$eq3 := x^2 = c^2\,(\xi^2 + 1)\,(-\eta^2 + 1)\cos(\varphi)^2$$

```
> eq4 := lhs(eq_y)^2 = rhs(eq_y)^2;
```

$$eq4 := y^2 = c^2\,(\xi^2 + 1)\,(-\eta^2 + 1)\sin(\varphi)^2$$

```
> eq5 := lhs(eq1)+lhs(eq3)+lhs(eq4) =
  factor(simplify(factor(rhs(eq1)+rhs(eq3)+rhs(eq4))));
```

$$eq5 := (z + I\,c)^2 + x^2 + y^2 = c^2\,(I\,\eta + \xi)^2$$

```
> eq6 := lhs(eq2)+lhs(eq3)+lhs(eq4) =
  factor(simplify(factor(rhs(eq2)+rhs(eq3)+rhs(eq4))));
```

$$eq6 := (z - I\,c)^2 + x^2 + y^2 = c^2\,(-I\,\eta + \xi)^2$$

```
> eq7 := sqrt(lhs(eq5),symbolic) = sqrt(rhs(eq5),symbolic);
```

$$eq7 := \sqrt{(z + I\,c)^2 + x^2 + y^2} = c\,(I\,\eta + \xi)$$

```
> eq8 := sqrt(lhs(eq6),symbolic) = sqrt(rhs(eq6),symbolic);
```

$$eq8 := \sqrt{(z - I c)^2 + x^2 + y^2} = c\,(-I\,\eta + \xi)$$

```
> eq9 := simplify(rhs(eq7)+rhs(eq8))=(lhs(eq7)+lhs(eq8));
```

$$eq9 := 2\,c\,\xi = \sqrt{(z + I c)^2 + x^2 + y^2} + \sqrt{(z - I c)^2 + x^2 + y^2}$$

```
> eq_xi := lhs(eq9)/coeff(lhs(eq9),xi) =
  rhs(eq9)/coeff(lhs(eq9),xi);
```

$$eq_xi := \xi = \frac{\sqrt{(z + I c)^2 + x^2 + y^2} + \sqrt{(z - I c)^2 + x^2 + y^2}}{2\,c}$$

```
> eq10 := simplify(rhs(eq7)-rhs(eq8))=(lhs(eq7)-lhs(eq8));
```

$$eq10 := 2\,I\,c\,\eta = \sqrt{(z + I c)^2 + x^2 + y^2} - \sqrt{(z - I c)^2 + x^2 + y^2}$$

```
> eq_eta := lhs(eq10)/coeff(lhs(eq10),eta) =
  rhs(eq10)/coeff(lhs(eq10),eta);
```

$$eq_eta := \eta = -\frac{\frac{1}{2}\,I\,(\sqrt{(z + I c)^2 + x^2 + y^2} - \sqrt{(z - I c)^2 + x^2 + y^2})}{c}$$

```
> eq_phi := phi =
  solve(convert(lhs(eq_y)/lhs(eq_x)=rhs(eq_y)/rhs(eq_x),tan),p
  i);
```

$$eq_phi := \varphi = \arctan\left(\frac{y}{x}\right)$$

```
>
```

# oblate to dipole:

## check converges to dipole as oblate spheroid becomes a sphere:

```
> restart;
```

```
> with(VectorCalculus):
```

```
Warning, the assigned names `<,>` and `<|>` now have a global
binding

Warning, these protected names have been redefined and
unprotected: `*`, `+`, `.`, D, Vector, diff, int, limit, serie
```

```
> assume(xi,real,eta,real,phi,real,theta,real,a,real,c,real);
```

```
> eq_x := x = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*cos(phi);
```

$$eq_x := x = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\cos(\varphi)$$

```
> eq_y := y = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*sin(phi);
```

$$eq_y := y = c\sqrt{(\xi^2+1)\,(-\eta^2+1)}\,\sin(\varphi)$$

```
> eq_z := z = c*eta*xi;
```

$$eq_z := z = c\,\eta\,\xi$$

as a=b the oblate spheroid becomes a sphere and the coordinate system goes to the standard spherical coordinate system:

```
> eq_x := x = convert(series(rhs(eq_x),xi=infinity,1),polynom)
```

$$eq_x := x = c\sqrt{-\eta^2+1}\,\cos(\varphi)\,\xi$$

```
> eq_x := simplify(subs(xi=r/c,eta=cos(theta),eq_x));
```

$$eq_x := x = \cos(\varphi)\,r\,|\sin(\theta)|$$

```
> eq_y := y = convert(series(rhs(eq_y),xi=infinity,1),polynom)
```

$$eq_y := y = c\sqrt{-\eta^2+1}\,\sin(\varphi)\,\xi$$

```
> eq_y := simplify(subs(xi=r/c,eta=cos(theta),eq_y));
```

$$eq_y := y = \sqrt{-\cos(\theta)^2+1}\,\sin(\varphi)\,r$$

```
> eq_z := simplify(subs(xi=r/c,eta=cos(theta),eq_z));
```

$$eq_z := z = \cos(\theta)\, r$$

write out equation for psi1:

```
> K := 1/2*(xi0^2+1)*c*Mx*xi0;
```

$$K := \left( \frac{1}{2}\, \xi 0^2 + \frac{1}{2} \right) c\, Mx\, \xi 0$$

```
> R := 1/2*(xi^2+1)^(1/2)*arctan(2*xi,
  -1+xi^2)-xi/(xi^2+1)^(1/2);
```

$$R := \frac{1}{2} \sqrt{\xi^2 + 1}\ \arctan(2\,\xi, \xi^2 - 1) - \frac{\xi}{\sqrt{\xi^2 + 1}}$$

```
> T := (-eta^2+1)^(1/2);
```

$$T := \sqrt{-\eta^2 + 1}$$

```
> P := cos(phi);
```

$$P := \cos(\varphi)$$

```
> psi1 := K*R*T*P;
```

$$\psi 1 := \left( \frac{1}{2}\, \xi 0^2 + \frac{1}{2} \right) c\, Mx\, \xi 0 \left( \frac{1}{2} \sqrt{\xi^2 + 1}\ \arctan(2\,\xi, \xi^2 - 1) - \frac{\xi}{\sqrt{\xi^2 + 1}} \right) \sqrt{-\eta^2 + 1}\ \cos($$

in the limit as a=b:

```
> R_ := series(R,xi=infinity,5);
```

$$R_ := \frac{2}{3\,\xi^2} + O\left( \frac{1}{\xi^4} \right)$$

```
> R_ := convert(R_,polynom);
```

$$R_ := \frac{2}{3\,\xi^2}$$

```
> T_ := simplify(subs(eta=cos(theta),T));
```

$$T_ := |\sin(\theta)|$$

M is the magnetization (in A/m), m is the magnetic moment (in A m^2) ; M=m/volume:

```
> K_ := simplify(subs(xi0=a/c,subs(Mx=mx/(4/3*pi*a^3),K)));
```

$$K_ := \frac{3\,(a^2 + c^2)\, mx}{8\, c^2\, \pi\, a^2}$$

```
> psi1 := simplify(subs(xi=r/c,K_*R_*T_*P));
```

$$\psi 1 := \frac{(a^2 + c^2)\, mx \;\; |\sin(\theta)| \;\; \cos(\varphi)}{4\,\pi\, a^2\, r^2}$$

```
> psi1 := simplify(subs(c=0,psi1));
```

$$\psi 1 := \frac{mx \;\; |\sin(\theta)| \;\; \cos(\varphi)}{4\,\pi\, r^2}$$

which, as will now be shown is the magnetic potential for an x-oriented point dipole in spherical coorinates:

```
> SetCoordinates(spherical[r,theta,phi]);
```

$$spherical_{r,\,\theta,\,\varphi}$$

the magnetic moment is:

```
> m := VectorField(<mx,0,0>,cartesian[x,y,z]);
```

$$m := mx\, \underline{e}_x$$

compute the scalar potenital of the dipole:

```
> psi := DotProduct(m,VectorField(<1,0,0>))/(4*pi*r^2);
```

$$\psi := \frac{mx \sin(\theta) \cos(\varphi)}{4\,\pi\, r^2}$$

## check converges to dipole at great distances:

```
> psi1 := K*R*T*P;
```

$$\psi 1 := \left( \frac{1}{2}\, \xi 0^2 + \frac{1}{2} \right) c\, Mx\, \xi 0 \left( \frac{1}{2} \sqrt{\xi^2 + 1}\, \arctan(2\,\xi, \xi^2 - 1) - \frac{\xi}{\sqrt{\xi^2 + 1}} \right) \sqrt{-\eta^2 + 1}\, \cos($$

```
> K_ := simplify(subs(Mx=mx/(4/3*pi*c^3*xi0*(xi0^2+1)),K));
```

$$K_ := \frac{3\, mx}{8\,\pi\, c^2}$$

R_, T_ and P are as before so:

```
> psi1 := simplify(subs(xi=r/c,K_*R_*T_*P));
```

$$\psi 1 := \frac{mx \;\; |\sin(\theta)| \;\; \cos(\varphi)}{4\,\pi\, r^2}$$

## oblate to prolate transformation:

### transform the coordinate system

```
> restart;
```

```
> eq_x := x = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*cos(phi);
```

$$eq_x := x = c\sqrt{(\xi^2 + 1)(1 - \eta^2)}\cos(\varphi)$$

```
> eq_y := y = c*sqrt( (xi^2+1)*(1-eta^2),symbolic )*sin(phi);
```

$$eq_y := y = c\sqrt{(\xi^2 + 1)(1 - \eta^2)}\sin(\varphi)$$

```
> eq_z := z = c*eta*xi;
```

$$eq_z := z = c\,\eta\,\xi$$

```
> assume(xi>=1,xi0>=1,eta<=1,eta>=-1,phi>0,phi<2*Pi,c>0);
```

```
> eq_x_p := subs(xi=-I*xi,c=I*c,eq_x):
```

```
> eq_x_p := lhs(eq_z_p) = sqrt(rhs(eq_x_p)^2,symbolic);
```

$$eq_x_p := z = c\cos(\varphi)\sqrt{(\xi^2 - 1)(1 - \eta^2)}$$

```
> eq_y_p := subs(xi=-I*xi,c=I*c,eq_y):
```

```
> eq_y_p := lhs(eq_y_p) = sqrt(rhs(eq_y_p)^2,symbolic);
```

$$eq_y_p := y = c\sin(\varphi)\sqrt{(\xi^2 - 1)(1 - \eta^2)}$$

```
> eq_z_p := subs(xi=-I*xi,c=I*c,eq_z);
```

$$eq_z_p := z = c\,\eta\,\xi$$

### transform the demagnetizing factor:

```
> assume(xi0,real,xi0>1);
```

```
> eq_d := d = 1/4*xi0*(xi0^2+1)*arctan(2*xi0,
  -1+xi0^2)-1/2*xi0^2;
```

$$eq_d := d = \frac{1}{4}\,\xi 0\,(\xi 0^2 + 1)\arctan\left(\frac{2\,\xi 0}{-1 + \xi 0^2}\right) - \frac{1}{2}\,\xi 0^2$$

```
> eq_d := subs(xi0=-I*xi0,eq_d);
```

$$eq_d := d = -\frac{1}{4}\,I\,\xi 0\,(-\xi 0^2 + 1)\arctan\left(-\frac{2\,I\,\xi 0}{-1 - \xi 0^2}\right) + \frac{1}{2}\,\xi 0^2$$

```
> eq_d := simplify(convert(eq_d,ln));
```

$$eq_d := d = \frac{1}{4}\,\xi 0\,(\xi 0^2\,\ln(\xi 0 - 1) - \xi 0^2\,\ln(\xi 0 + 1) - \ln(\xi 0 - 1) + \ln(\xi 0 + 1) + 2\,\xi 0)$$

```
> eq_d := combine(eq_d,ln,integer);
```

$$eq_d := d = \frac{1}{4}\,\xi 0\left(\xi 0^2\,\ln\left(\frac{\xi 0 - 1}{\xi 0 + 1}\right) + 2\,\xi 0 + \ln\left(\frac{\xi 0 + 1}{\xi 0 - 1}\right)\right)$$

```
> eq_d :=
  subs(ln((xi0+1)/(xi0-1))=LNX,ln((xi0-1)/(xi0+1))=-LNX,eq_d);
```

$$eq_d := d = \frac{1}{4}\,\xi 0\,(-\xi 0^2\,LNX + 2\,\xi 0 + LNX)$$

```
> eq_d := collect(eq_d,LNX);
```

$$eq_d := d = \frac{1}{4}\,\xi 0\,(-\xi 0^2 + 1)\,LNX + \frac{1}{2}\,\xi 0^2$$

```
> eq_d := subs(LNX=ln((xi0+1)/(xi0-1)),eq_d);
```

$$eq_d := d = \frac{1}{4}\,\xi 0\,(-\xi 0^2 + 1)\,\ln\left(\frac{\xi 0 + 1}{\xi 0 - 1}\right) + \frac{1}{2}\,\xi 0^2$$

which is the demagnetizing factor for a prolate spheroid.

## transform the scalar potential

write out equation for psi1:

```
> assume(xi0,real,xi0>1);
> K := 1/2*(xi0^2+1)*c*Mx*xi0:
> R := 1/2*(xi^2+1)^(1/2)*arctan(2*xi,
  -1+xi^2)-xi/(xi^2+1)^(1/2):
> T := (-eta^2+1)^(1/2):
> P := cos(phi):
> psi1 := K*R*T*P;
```

$$\psi 1 := \frac{1}{2}\,(\xi 0^2 + 1)\,c\,Mx\,\xi 0\left(\frac{1}{2}\sqrt{\xi^2 + 1}\,\arctan\left(\frac{2\,\xi}{\xi^2 - 1}\right) - \frac{\xi}{\sqrt{\xi^2 + 1}}\right)\sqrt{1 - \eta^2}\,\cos(\varphi)$$

```
> K := simplify(subs(xi0=-I*xi0,c=I*c,K));
```

$$K := -\frac{1}{2}\,(\xi 0^2 - 1)\,c\,Mx\,\xi 0$$

```
> R := expand(evalc(subs(xi=-I*xi,c=I*c,R)));
```

$$R := -\frac{1}{4}\sqrt{\xi^2 - 1}\,\ln\left(-\frac{2\,\xi}{-\xi^2 - 1} + 1\right) + \frac{1}{4}\sqrt{\xi^2 - 1}\,\ln\left(1 + \frac{2\,\xi}{-\xi^2 - 1}\right) + \frac{\xi}{\sqrt{\xi^2 - 1}}$$

```
> R := simplify(convert(R,ln));
```

$$R := -\frac{\ln(\xi + 1)\,\xi^2 - \ln(\xi + 1) - \ln(\xi - 1)\,\xi^2 + \ln(\xi - 1) - 2\,\xi}{2\sqrt{\xi^2 - 1}}$$

```
> R := combine(R,ln,integer);
```

$$R := -\frac{\xi^2 \ln\left(\frac{\xi + 1}{\xi - 1}\right) - 2\,\xi + \ln\left(\frac{\xi - 1}{\xi + 1}\right)}{2\sqrt{\xi^2 - 1}}$$

```
> R := subs(ln((xi+1)/(xi-1))=LNX,ln((xi-1)/(xi+1))=-LNX,R);
```

$$R := -\frac{\xi^2\,LNX - 2\,\xi - LNX}{2\sqrt{\xi^2 - 1}}$$

```
> R := collect(R,LNX);
```

$$R := -\frac{1}{2}\sqrt{\xi^2 - 1}\;LNX + \frac{\xi}{\sqrt{\xi^2 - 1}}$$

```
> R := subs(LNX=ln((xi+1)/(xi-1)),R);
```

$$R := -\frac{1}{2}\sqrt{\xi^2 - 1}\,\ln\left(\frac{\xi + 1}{\xi - 1}\right) + \frac{\xi}{\sqrt{\xi^2 - 1}}$$

```
> psi1 := K*R*T*P;
```

$$\psi 1 := -\frac{1}{2}\,(\xi 0^2 - 1)\,c\,Mx\,\xi 0\left(-\frac{1}{2}\sqrt{\xi^2 - 1}\,\ln\left(\frac{\xi + 1}{\xi - 1}\right) + \frac{\xi}{\sqrt{\xi^2 - 1}}\right)\sqrt{1 - \eta^2}\,\cos(\varphi)$$

```
>
>
```

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**TITLE: Transversely** Magnetized Oblate Spheroids (Sphéroïde Aplati Magnétisé Transversalement)

**PERSONAL AUTHOR:** Richards, T.C
**CORPORATE AUTHOR:** Defence R&D Canada - Atlantic, Dartmouth NS (CAN)

**ABSTRACT:** The use of magnetized spheroids has attracted considerable interest as a building block to model and predict the static magnetic fields around steel structures. Defence R&D Canada (DRDC) has made effective use of magnetized prolate spheroids for the near field magnetic modelling of naval ships as required for mine simulation studies. For some geometries however, oblate spheroids may provide a more realistic representation of the structure. In particular, presented here is the derivation of the scalar magnetic potential and magnetic fields arising from a transversely magnetized oblate spheroid. Additionally, the solution is verified to produce; the results for a sphere, a point dipole in the farfield and lastly a transversely magnetized prolate spheroid.
Il y a un résumé français ici.

**IDENTIFIERS:** magnetic modeling, static magnetic model, prolate spheroid, oblate spheroid, pro;spheroidal coordinates, oblate spheroidal coordinates, transversely magnetized

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|---|---|---|
| **REPORT NUMBER:** DRDC-ATLANTIC-SL-2005-136; Scientific Literature | **REPORT DATE:** 17 Jul 2005 | **NUMBER OF PAGES :** 33 |
| **DRDKIM N°:** CA026270 | **CANDIS N°:** 524091 | **Format(s):** Electronic Document(PDF) |

Date modified: 2005-04-05

Important Notices